Riesz Representation Theorem for Continuous Linear Functionals on a Hilbert Space

Let H be a H.S. and let 'f' be an arbitrary functional in H^*. Then $\exists$ a unique vector y in H s.t. $f = f_y$

i.e. $f(x) = f_y(x) = (x, y)$ $\forall x \in H$

Pf:- First we shall show if $\exists$ a vector y s.t. $f(x) = (x, y)$ $\forall x \in H$ —(1)

then y is unique.

Suppose y_1 & y_2 are two vectors satisfying this property.

Then we have—

$f(x) = (x, y_1)$ $\forall x \in H$

& $f(x) = (x, y_2)$ $\forall x \in H$

$\therefore$ we have —

$(x, y_1) = (x, y_2)$ $\forall x \in H$

$\Rightarrow (x, y_1 - y_2) = 0$ $\forall x \in H$

$\Rightarrow (y_1 - y_2, y_1 - y_2) = 0$, if we take $x = y_1 - y_2 \in H$

$\Rightarrow y_1 - y_2 = 0 \Rightarrow y_1 = y_2$

$\therefore y$ is unique.

Now we shall show that $\exists$ a vector y satisfying (1) $\forall x \in H$.

If 'f' is zero functional then

$$f(x) = 0 \quad \forall x \in H.$$

Also, if $y = 0$ then $(x, y) = (x, 0) = 0 \ \forall x \in H$

$\therefore$ If f is zero functional then the vector $y = 0$ is s.t. $f(x) = (x, y) \ \forall x \in H$

Now suppose that 'f' is not zero functional i.e. $f(x) \neq 0$ for some $x \in H$.

Let M be the null space of 'f' i.e.

$$M = \{x \in H ; f(x) = 0\}$$

Then M is a proper subspace of H.

Also, the null space of any continuous linear transformation is closed. Since f is continuous, $\therefore$ M is a proper closed subspace of H. $\therefore$ $\exists$ a non-zero vector $y_0 \in H$ s.t. $y_0 \perp M$ i.e. $y_0 \in M^{\perp}$.

We shall show that for some suitably choosen scalar α, the vector $y = \alpha y_0$ will serve our purpose.

First we observe that whatever may be the value of the scalar α, the vector $y = \alpha y_0$ satisfies (1), $\forall x \in M$. (34)

If $x \in M$ then $f(x) = 0$

Also if $x \in M$ then $(x, y) = (x, \alpha y_0)$

$$= \bar{\alpha}(x, y_0)$$

$$= 0$$

as $x \in M$ & $y_0 \perp M$

$$\Rightarrow (x, y_0) = 0$$

Thus if $x \in M$ and if $y = \alpha y_0$ then we have $f(x) = (x, y) = 0$ and (1) is satisfied.

Now let us try to choose 'α' in such a way that the vector $y = \alpha y_0$ satisfies (1) for $x = y_0$.

The condition this imposes on α is that -

$$f(y_0) = (y_0, \alpha y_0)$$

$$= \bar{\alpha}(y_0, y_0)$$

$$= \bar{\alpha} \|y_0\|^2$$

$\therefore$ if we take $\bar{\alpha} = \dfrac{f(y_0)}{\|y_0\|^2}$

or $\alpha = \dfrac{\overline{f(y_0)}}{\|y_0\|^2}$

then the vector $y = \alpha y_0$ satisfies (1) $\forall x \in M$ and for $x = y_0$.

Now we shall complete the proof by showing if a vector y satisfies (1) $\forall x \in M$ and for $x = y_0$ then it will satisfy (1) $\forall x \in H$.

Let x be an arbitrary vector in H. Since $M \cap M^{\perp} = \{0\}$ and y_0 is a non-zero vector belonging to $M^{\perp}$, $\therefore y_0 \notin M$. So we have—

$f(y_0) \neq 0$.

Now we can write $f(x) = \dfrac{f(x)}{f(y_0)} f(y_0)$

$= \beta f(y_0)$

where $\beta = \dfrac{f(x)}{f(y_0)}$

We have then—

$f(x) = \beta f(y_0) \Rightarrow f(x) - \beta f(y_0) = 0$

$\Rightarrow f(x - \beta y_0) = 0$; $\because f$ linear

$\Rightarrow x - \beta y_0 \in M$; M null space of f

$\Rightarrow x - \beta y = m$ for some $m \in M$

Now— $f(x) = f(m + \beta y_0)$

$= f(m) + \beta f(y_0)$; f linear

$= (m, y) + \beta (y_0, y)$, $\because f(x) = (x, y)$ $\forall x \in M$

$= (m, y) + (\beta y_0, y) = (m + \beta y_0, y) = (x, y)$

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Thus if a vector y satisfies (1) $\forall x \in M$ and for $x = y_0$ then it satisfies (1) $\forall x \in H$.

Hence $y = \alpha y_0$ is the required vector

where $\alpha = \dfrac{\overline{f(y_0)}}{\|y_0\|^2}$

(Proved)